PHYS 320 ANALYTICAL MECHANICS

Dr. Gregory W. Clark
Fall 2018

Calculus of Variations and Hamilton's Principle

$$J = \int_{x_1}^{x_2} f(y(x), y'(x); x)\,dx$$

where $y'(x) \equiv dy/dx$

has extreme values (is "stationary") when

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Similarly

$$S = \int_{t_1}^{t_2} L\,dt$$

where $L \equiv T - U$

Hamilton's principle

has extreme values (is stationary) when taken along the actual path, so that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

Lagrange's equation

where the q_i are generalized coordinates

Lagrangian Mechanics

Consider a system of n discrete particles …

- *Specifying* state of system *requires n position vectors,* $\vec{r}_n$
  - *thus,* 3n *quantities must be specified*
  - *not all of these are necessarily independent*
- *If ∃* m *equations of constraint (that relate various quantities), then ∃* s = (3n – m) *independent coordinates*
  - *no constraints? Fine! Then* s = 3n.

Lagrangian Mechanics

Generalized coordinates: any set of quantities that completely specifies the state of the system

$$q_1, q_2, q_3, \ldots, q_j$$

Generalized velocities: set of time derivatives of generalized coordinates

$$\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_j$$

Lagrangian Mechanics

Lagrangian function:

$$L \equiv T(q_j, \dot{q}_j, t) - U(q_j, t)$$

Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial q_j} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, 3, \ldots, s$$

Lagrangian Mechanics: single particle in 3D

Lagrangian: $L \equiv T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$

Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$$\frac{\partial L}{\partial y} = -\frac{\partial U}{\partial y} = F_y \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

$$\frac{\partial L}{\partial z} = -\frac{\partial U}{\partial z} = F_z \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\ddot{z}$$

$$\vec{F} = -\vec{\nabla} U = m\vec{a}$$